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AMPLITUDE DISTRIBUTION AND ENERGY BALANCE OF SMALL DISTURBANCES IN PLATE FLOW

By H. Schlichting

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AMPLITUDE DISTRIBUTION AND ENERGY BALANCE OF SMALL DISTURBANCES IN PLATE FLOW*

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SUMMARY

In a previous report by W. Tollmien, the stability of laminar flow past a flat plate was investigated by the method of small vibrations, and the wave length $\lambda = 2\pi/\alpha$, the phase velocity c_r, and the Reynolds number R of the neutral disturbances established. In connection with this, the present report deals with the average disturbance velocities $\sqrt{\overline{u'^2}}$ and $\sqrt{\overline{v'^2}}$ and the correlation coefficient $\frac{\overline{u'v'}}{\sqrt{\overline{u'^2} \times \overline{v'^2}}}$ as function of the wall distance y for two special neutral disturbances (one at the lower and one at the upper branch of the equilibrium curve in the αR plane). The maximum value of the last two quantities lies in the vicinity of the critical layer where the velocity of the basic flow and the phase velocity of the disturbance motion are equal. The energy balance of the disturbance motion is investigated. The transfer of energy from primary to secondary motion occurs chiefly in the neighborhood of the critical layer, while the dissipation is almost completely confined to a small layer next to the wall. The energy conversion in the two explored disturbances is as follows: In one oscillation period, half of the total kinetic energy of the disturbance motion on the lower branch of the equilibrium curve is destroyed by dissipation and replaced by the energy transferred from the primary to the secondary motion. For the disturbance on the upper branch of the equilibrium curve, about a fourth of the kinetic energy of the disturbance motion is dissipated and replaced in one oscillation period. The requirement that the total energy balance for the neutral disturbances be equal to zero is fulfilled with close approximation and affords a welcome check on the previous solution of the characteristic value problem.

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*"Amplitudenverteilung und Energiebilanz der kleinen Störungen bei der Plattenströmung." Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Neue Folge, Band 1, No. 4, 1935, pp. 47–78.

1. Introduction

The numerous efforts, within the last decade, to solve the problem of turbulence (reference 1) have at least produced some satisfactory results for a certain class of boundary-layer profiles when their stability was investigated by the method of small vibrations, with due consideration to the friction of the fluid and the profile curvature (reference 2). Referring to Tollmien (reference 3), who treated the example of laminar flow past a flat plate, the writer investigated several other cases: the Couette flow (reference 4), the amplification of the unstable disturbances in plate flow (reference 5), and the stabilizing effect of a stratification by centrifugal forces (reference 6), and temperature gradients (reference 7). Every one of the investigations was restricted to the solution of the corresponding characteristic-value problems, without calculating the characteristic function itself. In that manner, the wave lengths of the unstable, hence "dangerous", disturbances were identified as function of the Reynolds number. In most cases, only the disturbances situated right at the boundary, between amplification and damping, were determined. For these, just as much energy is transferred from the primary to the secondary flow, as secondary-motion energy is dissipated by the friction so that the total energy balance is zero.

All the stability studies made up to now were, for reasons of mathematical simplicity, based upon an assumedly plane fundamental flow, which depends only on the coordinate transverse to the direction of the flow, and a plane superposed disturbance motion which propagates in form of a wave motion in the primary-flow direction. While there is no objection to the limitation to the plane fundamental flow, since it is frequently realized experimentally, objection may be raised to the plane disturbance motion because the disturbances accidentally produced in practice are almost always three-dimensional. Accordingly, it might appear as if the limitation to two-dimensional disturbances was all too special. However, H. B. Squire (reference 8) recently demonstrated on the Couette flow – this theorem is equally applicable to boundary-layer profiles – that precisely the specific case of the two-dimensional disturbance motion is particularly suitable for the stability study in the following sense: According to Squire, a two-dimensional flow, which is unstable against three-dimensional disturbances at a certain Reynolds number, is unstable against two-dimensional disturbances even at a lower Reynolds number. The two-dimensional disturbances are therefore "more dangerous" for a flow than the three-dimensional. The critical Reynolds number, which is defined as lowest stability limit, is thus obtained precisely from the two-dimensional, not the three-dimensional, disturbances.

To gain a deeper insight into the mechanism of the turbulence phenomena from small unstable disturbances, a more detailed knowledge of the properties of these small disturbances is necessary. The present report, therefore, deals first, with the distribution of the amplitude of the disturbance over the flow section, that is, the calculation of the characteristic functions and second, with the study of the energy distribution and energy balance of the disturbance motion. The investigations are based upon the disturbances of the laminar flow past a flat plate which are situated exactly at the boundary between amplification and damping (neutral oscillations).

Chapter I

AMPLITUDE DISTRIBUTION

2. Discussion of the differential equation of disturbance.

Let $U(y)$ be the velocity distribution of the fundamental flow (fig. 1) and ψ the flow function of the superposed disturbance motion, which is assumed as a wave motion moving in the x direction (direction of primary flow), whose amplitude φ is solely dependent on y, hence

$$\psi(x,y,t) = \varphi(y)e^{i(\alpha x-\beta t)} = \varphi(y)e^{i\alpha(x-ct)}$$

α is real and $\lambda = 2\pi/\alpha$ is the wave length of the disturbance; $\beta = \beta_r + i\beta_i$ and $c = \beta/\alpha$ are, in general, complex; $T = 2\pi/\beta_r$ is the period of oscillation; β_i indicates the amplification or the damping, depending upon whether positive or negative; $c_r = \beta_r/\alpha$ is the phase velocity of the disturbance. For the disturbance amplitude φ, after introduction of dimensionless variables from the Navier-Stokes differential equations, it results in a linear-differential equation of the fourth order, the differential equation of the disturbance

$$(U - c)(\varphi'' - \alpha^2\varphi) - U''\varphi = -\frac{i}{\alpha R}(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) \qquad (1)$$

($R = U_m\delta/\nu$ = Reynolds number, U_m = constant velocity outside of the boundary layer, δ = characteristic length of the boundary-layer profile = boundary-layer thickness, ν = kinematic viscosity.) The general solution φ of the disturbance equation is built up from four particular solutions $\varphi_1, \ldots, \varphi_4$

$$\varphi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + C_4\varphi_4 \qquad (2)$$

The boundary conditions $\varphi = \varphi' = 0$ for $y = 0$ and $y = \infty$ give (as explained in the report cited in reference (5))

$$C_4 = 0 \tag{3a}$$

and for C_1, C_2, C_3 the system of equations

$$\left.\begin{aligned}
&C_1\varphi_{10} + C_2\varphi_{20} + C_3\varphi_{30} = 0\\
&C_1\varphi_{10}' + C_2\varphi_{20}' + C_3\varphi_{30}' = 0\\
&C_1\Phi_{1a} + C_2\Phi_{2a} = 0\\
&\left(\Phi_{\nu a} = \varphi_{\nu a}' + \alpha\varphi_{\nu a};\ \nu = 1,\ 2\right)
\end{aligned}\right\} \tag{4}$$

The subscript 0 indicates the values at the wall $y = 0$, subscript a the values in the connecting point $y = a$ to the region of constant velocity. From (4) the equation of the characteristic value problem follows as

$$\begin{vmatrix} \varphi_{10} & \varphi_{20} & \varphi_{30} \\ \varphi_{10}' & \varphi_{20}' & \varphi_{30}' \\ \Phi_{1a} & \Phi_{2a} & 0 \end{vmatrix} = 0 \tag{5}$$

This equation is discussed in the earlier reports for several cases. It contains, aside from the constants of the basic profile, the parameters α, R, c_r, and c_i. The complex equation (5) is equivalent to two real equations, and, if limited to the case of neutral disturbances $\left(c_i = 0\right)$, these two equations give, after elimination of c_r, one equation between α and R. This is the equation of the neutral curve in the αR plane, which separates the unstable from the stable disturbance attitudes, and was originally computed by Tollmien for the plate flow. They are assumed to be known for the subsequent study (fig. 2).[1]

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[1]The writer computed the neutral curve for the plate flow cited under reference 5 again and found some differences with respect to Tollmien's report. The newly obtained values are used in this study.

In the following, the amplitude and energy distribution is computed for two neutral oscillations, one of which lies on the lower, the other on the upper branch of the equilibrium curve (fig. 2). The parameters of these two neutral oscillations, obtained from the earlier calculation, are indicated in table 1.

For the calculation of the integration constants C_1, C_2, C_3 from (4), we put

$$C_1 = 1 \tag{3b}$$

because the amplitude of disturbance remains indeterminate up to a constant factor, the intensity factor of the disturbance motion. Thus, for the other two constants

$$\left.\begin{aligned} C_2 &= -\frac{\Phi_{1a}}{\Phi_{2a}} \\ C_3 &= \frac{1}{\varphi_{30}}\left\{\frac{\Phi_{1a}}{\Phi_{2a}}\varphi_{20} - \varphi_{10}\right\} = \frac{1}{\varphi_{30}'}\left\{\frac{\Phi_{1a}}{\Phi_{2a}}\varphi_{20}' - \varphi_{10}'\right\} \end{aligned}\right\} \tag{6}$$

The particular solutions $\varphi_1(y)$ and $\varphi_2(y)$ are readily obtainable by expansion in series from the so-called frictionless differential equation of disturbance of the second order, which follows from the general equation (1) by omission of the terms on the right-hand side afflicted with the small factor $1/\alpha R$. The point $U = c_r$, where phase velocity of disturbance motion and basic flow velocity agree, and which is termed the critical point $y = y_k$, is a singular point of the frictionless differential equation of disturbance, which plays a prominent part in the investigations.

Putting

$$\frac{y - y_k}{y_k} = y_1; \ \frac{y - y_k}{a - y_k} = y_2; \ \alpha y_k = \alpha_1; \ \alpha(a - y_k) = \alpha_2 \tag{7}$$

where subscript k denotes the values at the critical points, the frictionless solutions for linear[2] velocity distribution read

$$\varphi_1 = \alpha^{-1} \sinh(\alpha_1 y_1); \ \varphi_2 = \cosh(\alpha_1 y_1) \tag{8}$$

and for parabolic velocity distribution

$$\frac{\varphi_1}{a - y_k} = a_1 y_2 + a_2 y_2^2 + a_3 y_2^3 + \cdots$$

$$\varphi_2 = b_0 + b_1 y_2 + b_2 y_2^2 + \cdots + \frac{U_k''}{U_k'} \varphi_1 \log y_2 \quad \text{for} \quad y_2 > 0 \tag{9}$$

$$\varphi_2 = b_0 + b_1 y_2 + b_2 y_2^2 + \cdots + \frac{U_k''}{U_k'} \varphi_1 \left(\log|y_2| - i\pi\right) \quad \text{for} \quad y_2 < 0$$

---

[2]The Blasius profile of the plate flow is approximated by a linear and a quadratic function (fig. 1), namely

$$0 \leqq y/\delta \leqq 0.175: \quad U/U_m = 1.68\, y/\delta$$

$$0.175 \leqq y/\delta \leqq 1.015: \quad U/U_m = 1 - (1.015 - y/\delta)^2 \tag{7a}$$

$$y/\delta \geqq 1.015: \quad U/U_m = 1$$

For the connection between δ and the displacement thickness δ^* used in figure 2, $\delta^* = 0.341\delta$ is applicable.

According to earlier data, the coefficients are given by the equations

$$
\left.
\begin{aligned}
& a_1 = 1;\ a_2 = -\frac{1}{2};\ a_3 = \frac{\alpha_2^{\,2}}{6};\ a_4 = -\frac{\alpha_2^{\,2}}{18}, \\
& a_5 = -0.0013\ \alpha_2^{\,2} + 0.0083\ \alpha_2^{\,4};\ a_6 = 0.0024\ \alpha_2^{\,2} + 0.0006\ \alpha_2^{\,4} \\
& b_0 = 1;\ b_1 = 0;\ b_2 = -1 + \frac{\alpha_2^{\,2}}{2} \\
& b_3 = 0.125 + 0.056\ \alpha_2^{\,2};\ b_4 = 0.021 - 0.141\ \alpha_2^{\,2} + 0.042\ \alpha_2^{\,4} \\
& b_5 = 0.005 + 0.005\ \alpha_2^{\,2} + 0.004\ \alpha_2^{\,4} \\
& b_6 = 0.0015 + 0.0012\ \alpha_2^{\,2} - 0.0038\ \alpha_2^{\,4} + 0.0014\ \alpha_2^{\,6}
\end{aligned}
\right.
\tag{10}
$$

The particular solution φ_1, with its derivatives, is regular throughout the entire range of flow $\left(-1 \leqq y_1 \leqq 0,\ 0 \leqq y_2 \leqq +1\right)$, and can be numerically computed with these data. But the particular solution φ_2 has a singularity, in which $\varphi_2{}'$ becomes logarithmically infinite in the critical layer $y = y_k$. The more detailed discussion has shown that the friction at the wall, and in a restricted vicinity of the critical layer, must be taken into consideration. The first gives the friction solution φ_3, the other, the friction correction for φ_2. Introducing the new variable

$$
\left.\eta = \left(y - y_k\right) \left(\alpha R U_k{}'\right)^{1/3} = \frac{y - y_k}{\epsilon}\right\} \tag{11}
$$

gives (only the greatest terms from (1) for $\varphi(\eta)$ being taken into account) the differential equation

$$
i\varphi'''' + \eta\varphi'' = \epsilon\frac{U_k{}''}{U_k{}'} \tag{12}
$$

from which follows the correction for φ_2 near the critical layer, as well as the third friction-affected solution φ_3. The calculation of these two solutions merits a little closer study.

3. The friction solution φ_3

The friction solution φ_3 is obtained from the differential equation (12) when the inhomogeneous terms encumbered with the small factor ϵ are omitted; hence, from the differential equation

$$i\varphi_3'''' + \eta\varphi_3'' = 0$$

Unusual in this equation is that, in contrast to the complete disturbance equation (1) and to the frictionless equation of disturbance, the dependence of the parameters α, R, and U' by (11) enters only as scale factor for y, and that it is not at all affected by U". As a result, $\varphi_3(\eta)$ can be computed once for all entirely independent from the velocity profile. In this instance, Tietjens' report (reference 9) constitutes a valuable support. A fundamental system $\varphi_{31}'' = F_1''$ and $\varphi_{32}'' = F_2''$ of equation (13) is given by

$$\eta^{1/2}\, J_{1/3}\left[\frac{2}{3}\eta^{3/2}\, e^{-i\pi/4}\right] \quad \text{and} \quad \eta^{1/2}\, J_{-1/3}\left[\frac{2}{3}\eta^{3/2}\, e^{-i\pi/4}\right] \tag{14}$$

The expansion in series of the Bessel functions

$$J_p(z) = \left(\frac{z}{2}\right)^p \sum_{\nu=0}^{\infty} \frac{(iz/2)^{2\nu}}{\nu!\Gamma(p + \nu + 1)}$$

gives, when the constant factors $\left(\frac{1}{3}\right)^{1/3} (-i)^{1/6} / \Gamma\left(\frac{4}{3}\right)$

and $\left(\frac{1}{3}\right)^{-1/3} (-i)^{-1/6} / \Gamma\left(\frac{2}{3}\right)$ are omitted

$$\left.\begin{aligned}
F_1''(\eta) &= \eta - \frac{\eta^7}{2!3^2 \cdot 4 \cdot 7} + \frac{\eta^{13}}{4!3^4 \cdot 4 \cdot 7 \cdot 10 \cdot 13} - \frac{\eta^{19}}{6!3^6 \cdot 4 \cdot 7 \cdot 10 \cdot 13 \cdot 16 \cdot 19} + - \cdots \\
&\quad + i \left\{ \frac{\eta^4}{1!3^1 \cdot 4} - \frac{\eta^{10}}{3!3^3 \cdot 4 \cdot 7 \cdot 10} + \frac{\eta^{16}}{5!3^5 \cdot 4 \cdot 7 \cdot 10 \cdot 13 \cdot 16} - + \cdots \right\} \\
F_2''(\eta) &= 1 - \frac{\eta^6}{2!3^2 \cdot 2 \cdot 5} + \frac{\eta^{12}}{4!3^4 \cdot 2 \cdot 5 \cdot 8 \cdot 11} - + \cdots \\
&\quad + i \left\{ \frac{\eta^3}{1!3^1 \cdot 2} - \frac{\eta^9}{3!3^3 \cdot 2 \cdot 5 \cdot 8} + \frac{\eta^{15}}{5!3^5 \cdot 2 \cdot 5 \cdot 8 \cdot 11 \cdot 14} - + \cdots \right\}
\end{aligned}\right\} \quad (15)$$

Owing to the boundary condition $\varphi_3'' = 0$ for $\eta = +\infty$ only the solution aggregate $\beta_1 F_1'' + \beta_2 F_2''$ approaching zero for great positive real η (β_1, β_2 = integration constants) comes into question. For great η, this can be represented by the Hankel function of the second kind with the subscript $\frac{1}{3}$ and the argument $\frac{2}{3}\eta^{3/2} e^{-i\pi/4}$, hence by

$$\varphi_3'' = F_4''(\eta) = \eta^{1/2} H_{1/3}^{(2)} \left[\frac{2}{3}\eta^{3/2} e^{-i\pi/4}\right]$$

Herewith, the looked-for solution of (13) — the constant factor being put as $\varphi_3'(\eta_o) = \varphi_{30}'$ for the sake of simplicity[3] — is given as expansion in power series near $\eta = 0$

$$\frac{\varphi_3''}{\varphi_{30}'} = \left(\beta_1 F_1'' + \beta_2 F_2''\right) \tag{16a}$$

and as asymptotic expansion for great η

$$\frac{\varphi_3''}{\varphi_{30}'} = BF_4'' \quad \left(B\varphi_{30}' = 1\right)$$

Integrating between the limits η_o and η gives

$$\frac{\varphi_3'}{\varphi_{30}'} = \beta_1 \int_{\eta_o}^{\eta} F_1'' d\eta + \beta_2 \int_{\eta_o}^{\eta} F_2'' d\eta + 1 \tag{16b}$$

$$\frac{\varphi_3}{\varphi_{30}'} = \beta_1 \int_{\eta_o}^{\eta}\int_{\eta_o}^{\eta} F_1'' d\eta d\eta + \beta_2 \int_{\eta_o}^{\eta}\int_{\eta_o}^{\eta} F_2'' d\eta d\eta + \eta - \eta_o - D \tag{16c}$$

where B and D are additional integration constants dependent on η_o; η_o is the η coordinate at the wall, $y = 0$; hence by (11)

$$\eta_o = -y_k/\epsilon = -y_k\left(\alpha R U_k'\right)^{1/3} \tag{11a}$$

The boundary condition $\varphi_3'/\varphi_{30}' = 1$ at $y = 0$, that is, $\eta = \eta_o$, is fulfilled by adding the term 1 in equation (16b).

---

[3] $-C_3\varphi_{30}'$ is the gliding speed of the frictionless solution at the wall.

Tietjens computed the integration constants β_1, β_2, B, D from the fact that at a point $\eta = \eta_1$, which lies in the range of validity of both expansions, asymptotic expansion and power series expansion of the solution of (13) up to and including the third differential quotient must agree. The resultant equation system, set up and solved by Tietjens, reads

$$\left.\begin{aligned}
&\beta_1 F_1''' + \beta_2 F_2''' = BF_4''' && \text{(a)}\\
&\beta_1 F_1'' + \beta_2 F_2'' = BF_4'' && \text{(b)}\\
&\beta_1 \int_{\eta_o}^{\eta} F_1''d\eta + \beta_2 \int_{\eta_o}^{\eta} F_2''d\eta + 1 = BF_4' && \text{(c)}\\
&\beta_1 \int_{\eta_o}^{\eta}\int_{\eta_o}^{\eta} F_1''d\eta d\eta + \beta_2 \int_{\eta_o}^{\eta}\int_{\eta_o}^{\eta} F_2''d\eta d\eta + \eta - \eta_o - D = BF_4 && \text{(d)}
\end{aligned}\right\} \quad (17)$$

The integration constants β_1, β_2, B, and D can also be computed by a simpler method than Tietjens', with the aid of the transition formula from Hankel's to Bessel's functions, which reads[4]

$$H_{1/3}^{(2)}(z) = -\frac{i}{\sin \pi/3}\left\{e^{\pi i/3} J_{1/3}(z) - J_{-1/3}(z)\right\}$$

This obviates the joining of the two expansions, makes (17) superfluous, and (17b) gives the exact values immediately:

$$\frac{\beta_2}{\beta_1} = -e^{-i\pi/6}\, 3^{2/3} \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \quad \text{and} \quad \frac{B}{\beta_1} = e^{i\pi/4}\, 3^{1/3}\, \Gamma\left(\frac{4}{3}\right) \sin\frac{\pi}{3} \qquad (18)$$

$$= -1.190 + 0.687i \qquad\qquad = 0.789(1 + i)$$

---

[4]Tollmien, who pointed this out to the writer, had this representation as far back as 1929, but summarily took over the data by Tietjens for the sake of simplicity.

The factors β_1 and D as function of η_o can then also be indicated explicitly, namely

$$\left.\begin{aligned} \beta_1 &= \left\{\left(1 - e^{i\pi/3}\right)3^{1/3}\,\Gamma\left(\frac{4}{3}\right) + \int_0^{\eta_o} F_1''d\eta + \frac{\beta_2}{\beta_1}\int_0^{\eta_o} F_2''d\eta\right\}^{-1} \\ D &= -\eta_o + \beta_1\left\{\int_{\eta_o}^{0}\int_{\eta_o}^{\eta} F_1''d\eta d\eta + \frac{\beta_2}{\beta_1}\int_{\eta_o}^{0}\int_{\eta_o}^{\eta} F_2''d\eta d\eta\right\} - B\varphi_3(0) \end{aligned}\right\} \quad (19)$$

with

$$B\varphi_3(0) = i\beta_1$$

Table 2 contains the results of a new calculation of Tietjens' value carried out by these formulas. The differences from Tietjens' figures are insignificant. The values of F_1'', F_2'', $\int_0^{\eta} F_1''d\eta = F_1'$, $\int_0^{\eta} F_2''d\eta = F_2'$, $\int_0^{\eta} F_1'd\eta = F_1$, $\int_0^{\eta} F_2'd\eta = F_2$ as function of η are indicated separately as real and imaginary part in table 3. Since, according to (15), these quantities are either symmetrical or antisymmetric functions of η, this table can be continued immediately according to the negative values of η.

For the two neutral oscillations, whose amplitude and energy distribution is to be computed, it is

$$\eta_o = -2.63 \quad \text{and} \quad \eta_o = -4.05$$

The corresponding values of ϵ, according to (11a), are given in table I. along with the integration constants β_1, β_2, D obtained by interpolation for these η_o values.

This takes care of all the data necessary for computing the friction solution φ_3 with its first and second derivative as function of η by the equations (16a, b, and c). Table 4 gives the thus obtained values of $\frac{\varphi_3''}{\varphi_{30}'}\frac{\varphi_3'}{\varphi_{30}'}\frac{\varphi_3}{\varphi_{30}'}$ as function of η. The connection between η and y is, according to equation (11),

$$y = y_k + \epsilon\eta$$

In all cases, the friction solution φ_3 from the wall toward the inside of the flow is very quickly damped out; but it still extends a little beyond the critical layer for both oscillations.

4. The friction correction of φ_2 in the intermediate layer

The second frictionless solution φ_2 behaves singularly at the critical layer $y = y_k$, namely through equation (9) as $\frac{U_k''}{U_k'}(y - y_k)\log(y - y_k)$, so that φ_2' behaves as $\frac{U_k''}{U_k'}\left\{1 + \log(y - y_k)\right\}$ and φ_2'' behaves as $\frac{U_k''}{U_k'}\frac{1}{(y - y_k)}$.

From the differential equation (12), in which only the greatest friction terms are taken into account, follows a solution φ_2 modified by the friction, which joins the frictionless solution at some distance from the critical point. For this purpose φ_2 is expanded in powers of the previously introduced small quantity $\epsilon = (\alpha RU_k')^{-1/3}$

$$\varphi_2 = \varphi_{20} + \epsilon\varphi_{21} + \cdots \tag{20}$$

φ_{20} being chosen equal to unity. From (12) follows the inhomogeneous differential equation

$$\varphi_{21}'''' - i\eta\varphi_{21}'' = -i\frac{U_k''}{U_k'} \tag{21}$$

for φ_{21} with reference to η.

On account of the very small value of ϵ, η can assume great values even at small $y - y_k$ values. An attempt is made to find such a solution φ_{21}'' of this equation, which for large η, but small $(y - y_k)$ joins on to the frictionless solution

$$\frac{d^2\varphi_2}{dy^2} = \frac{U_k''}{U_k'}\frac{1}{y - y_k}$$

For large η there shall be:

$$\frac{d^2\varphi_{21}}{dy^2} = \frac{U_k''}{U_k'\eta} \tag{22}$$

The corresponding homogeneous equation appeared earlier in the calculation of φ_3 (equation (13)). It has the fundamental system $F_1''(\eta)$ and $F_2''(\eta)$ (equation (15)). A particular solution of (21) is

$$\varphi_{21}''(\eta) = -i\,\frac{U_k''}{U_k'}\left\{F_1''\int_0^\eta F_2''d\eta - F_2''\int_0^\eta F_1''d\eta\right\}$$

which can be verified easily by substitution, and the general solution of (21) is

$$\varphi_{21}''(\eta) = \frac{U_k''}{U_k'}\left\{iF_2''\int_0^\eta F_1''d\eta - iF_1''\int_0^\eta F_2''d\eta + c_1F_1'' + c_2F_2''\right\} \tag{23}$$

The integration constants c_1 and c_2, which can be complex, are evaluated from the boundary conditions. The quantity φ_{21}'' is complex and shall join the real frictionless value (equation (22)) for large values of η. By decomposition in real and imaginary parts, the four equations defining the integration constants read

$$\frac{U_k'}{U_k''}\varphi_{21r}'' = G''(\eta) = -F_{2i}''F_{1r}' - F_{2r}''F_{1i}' + F_{1i}''F_{2r}' + F_{1r}''F_{2i}'$$

$$+ c_{1r}F_{1r}'' - c_{1i}F_{1i}'' + c_{2r}F_{2r}'' - c_{2i}F_{2i}'' = \frac{1}{\eta}$$

$$\frac{U_k'}{U_k''}\varphi_{21i}'' = H''(\eta) = - F_{2i}''F_{1i}' + F_{2r}''F_{1r}' + F_{1i}''F_{2i}' - F_{1r}''F_{2r}' \qquad (24)$$

$$+ c_{1r}F_{1i}'' + c_{1i}F_{1r}'' + c_{2r}F_{2i}'' + c_{2i}F_{2r}'' = 0$$

$$\text{for} \quad \eta = \pm\eta_1$$

From (21), with the boundary condition (22), it follows that φ_{21r}'' is an antisymmetrical, and φ_{21i}'' a symmetrical function of η. Moreover, since F_{1r}', F_{2i}', F_{2r}'', F_{1i}'' are symmetrical and F_{1i}', F_{2r}', F_{2i}'', F_{1r}'' antisymmetrical functions of η, the following must be true

$$c_{1i} = c_{2r} = 0 \qquad (25a)$$

for reasons of symmetry. The other two constants c_{1r}, c_{2i} are obtained by solving the above equation system for $\eta = \eta_1$. For the present calculation, $\eta_1 = 4$ was chosen. The series for the Bessel functions are still fairly convergent for $\eta = 4$; but since differences of very large numbers occur, the separate terms in (24) must be computed to five digits (table 3). For c_{1r} and c_{2i}

$$c_{1r} = 1.2852; \; c_{2i} = 0.9373 \qquad (25b)$$

so that $\frac{U_k'}{U_k''}\varphi_{21r}'' = G''(\eta)$ and $\frac{U_k'}{U_k''}\varphi_{21i}'' = H''(\eta)$ can be calculated. The values are given in table 5.

The values of φ_2 and $\frac{d\varphi_2}{dy}$ in the intermediate layer are obtained immediately by quadratures, namely

$$\frac{d\varphi_{2r}}{dy} = \frac{U_k''}{U_k'}\left\{(1 + \log \epsilon) + G'(\eta)\right\} \tag{26a}$$

and

$$\frac{d\varphi_{2i}}{dy} = \frac{U_k''}{U_k'}H'(\eta) = \frac{U_k''}{U_k'}\int_{\eta=4}^{\eta} H''(\eta)\,d\eta = \frac{U_k''}{U_k'}H'(\eta) \tag{26b}$$

A check on this numerical calculation is given by the fact that for $\frac{d\varphi_{2i}}{dy}$ at transition from large positive to large negative η the transition substitution for φ_2 deduced by Tollmien (reference 3) must result again (compare equation (9)), which he obtained by discussion of the asymptotic representation of the Hankel functions. Tollmien's transition substitution gives

$$\left(\frac{d\varphi_{2i}}{dy}\right)_{y=+\infty} = \left(\frac{d\varphi_{2i}}{dy}\right)_{y=-\infty} + \frac{U_k''}{U_k'}\pi$$

the present numerical calculation gives

$$\left(\frac{d\varphi_{2i}}{dy}\right)_{y=+\infty} = \left(\frac{d\varphi_{2i}}{dy}\right)_{y=-\infty} + \frac{U_k''}{U_k'}\int_{-\infty}^{+\infty} H''(\eta)\,d\eta$$

and the graphical integration gives

$$\int_{-\infty}^{+\infty} H''(\eta)\,d\eta \approx \int_{-4}^{+4} H''(\eta)\,d\eta = 3.14 \tag{27}$$

that is, complete agreement within the scope of mathematical accuracy.

The intermediate layer near the critical point, which by the present calculation reaches from about $\eta = -4$ to $\eta = +4$, is already so wide at the first neutral oscillation that it reaches up to the wall (wall $\eta_o = -2.63$); at the second oscillation with $\eta_o = -4.05$, the boundary of the intermediate layer is reached exactly at the wall. With this, all data needed for the numerical calculation of the solution φ_2, corrected by the friction with φ_2' and φ_2'', are available.

5. The numerical values of the integration constants

All three particular solutions φ_1, φ_2, φ_3 are numerically known. To build up the required solution φ from it, the numerical values of the integration constants C_2 and C_3 must be accertained (equation (6)). First of all, equation (2) is rewritten in a more suitable form, namely

$$\varphi = \varphi_1 + C_2\varphi_2 + C_3' \frac{\varphi_3}{\varphi_{30}'} \tag{2a}$$

where equations (3a) and (3b) were resorted to and φ_3 was replaced by the quantity φ_3/φ_{30}' which follows immediately from the numerical calculations. Comparison with (6) gives

$$C_3' = \frac{\Phi_{1a}}{\Phi_{2a}}\varphi_{20}' - \varphi_{10}' = -\left(C_2\varphi_{20}' + \varphi_{10}'\right) \tag{6a}$$

This method of writing has the advantage that the two integration constants C_2 and C_3' in (2a) are dependent only on the values of the frictionless solutions φ_1 and φ_2, hence are relatively simple to compute.

The values of φ_{1a}, φ_{2a}, φ_{1a}', φ_{2a}', Φ_{1a}, Φ_{2a} and the values of C_2 and C_3' thus computed by (6) and (6a) for both neutral oscillations are given in table 1. Table 6 and figures 3 and 4 give the values of φ_r, φ_i, φ_r', φ_i' computed with it, hence the desired amplitude distribution as function of y/δ.

Outside of the boundary layer, at $y/\delta > 1.015$ the simple formula

$$\left.\begin{aligned} \varphi_r &= C^* e^{-\alpha y} \qquad \varphi_r' = -\alpha C^* e^{-\alpha y} \\ \varphi_i &= \varphi_i' = 0 \end{aligned}\right\} \tag{29}$$

is valid for the amplitude distribution. The constant C^* is so chosen that the value of φ_r' joins the already found value in $y/\delta = 1.015$. (Table 1.)

6. The average fluctuation velocities and the correction factor

(compare reference 10)

Changing to the real method of writing

$$\left.\begin{aligned} u' &= \frac{\partial\psi}{\partial y} = K\left\{\varphi_r' \cos(\alpha x - \beta_r t) - \varphi_i' \sin(\alpha x - \beta_r t)\right\} U_m \\ v' &= -\frac{\partial\psi}{\partial x} = K\alpha\left\{\varphi_r \sin(\alpha x - \beta_r t) + \varphi_i \cos(\alpha x - \beta_r t)\right\} U_m \end{aligned}\right. \tag{30}$$

K is a freely available intensity factor. According to figures 3 and 4, the one phase (φ_r or φ_r') predominates in both neutral oscillations.

The amplitude distribution of u' and v' can be represented most appropriately by forming, in analogy with the turbulent fluctuation velocity, the dimensionless quantities $\frac{\sqrt{\overline{u'^2}}}{U_m}$ and $\frac{\sqrt{\overline{v'^2}}}{U_m}$, where the dash denotes the time average value formation over a period T at a fixed point x, y, or in other words

$$\overline{u'^2} = \frac{1}{T}\int_{t=0}^{T} u'^2 dt \quad (T = \text{vibration period})$$

The result is

$$\frac{\sqrt{\overline{u'^2}}}{U_m} = \frac{K}{\sqrt{2}}\sqrt{\varphi_r'^2 + \varphi_i'^2}; \quad \frac{\sqrt{\overline{v'^2}}}{U_m} = \frac{K\alpha}{\sqrt{2}}\sqrt{\varphi_r^2 + \varphi_i^2} \tag{31}$$

and

$$\frac{\overline{u'^2} + \overline{v'^2}}{U_m^2} = \frac{K^2}{2}\left\{\varphi_r'^2 + \varphi_i'^2 + \alpha^2(\varphi_r^2 + \varphi_i^2)\right\} \tag{32}$$

The last quantity gives the mean kinetic energy of the motion disturbance. (See eq. 36.) These averages, which are independent of x, are represented in figures 5, 6, and 7 and table 7 for both neutral vibrations as functions of y/δ. The intensity factor itself was so chosen that the average value of $\sqrt{\overline{u'^2}}$ in the boundary layer is equal to $0.05U_m$ $\left(\frac{1}{\delta}\int_0^\delta \sqrt{\overline{u'^2}}\,dy = 0.05U_m\right)$ (table 1). The maximum amplitude for both neutral vibrations lies near the critical layer. The correlation factor between u' and v', which is completely independent from the intensity of the motion disturbance, can then also be calculated. It is

$$k(u', v') = \frac{1}{\sqrt{\overline{u'^2}\ \overline{v'^2}}}\frac{1}{T}\int_0^T u'v'dt = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}\ \overline{v'^2}}}$$

$$k(u', v') = \frac{\varphi_r'\varphi_i - \varphi_r\varphi_i'}{\sqrt{(\varphi_r'^2 + \varphi_i'^2)\ (\varphi_r^2 + \varphi_i^2)}} \tag{33}$$

The correlation factor is likewise dependent on y/δ only; its variation is indicated in figure 8 and table 7. It is negative almost throughout the entire range of the flow, for both neutral vibrations, as is to be expected, since, owing to the positive dU/dy, positive u' is usually coupled with negative v' and negative u' with positive v'. The maximum value of k is −0.17 and −0.19, respectively. It is interesting to compare the theoretically established correlation coefficient with Townend's data in a developed turbulent flow (reference 11). The k values of −0.16 to −0.18, obtained for the flow in a channel of square cross section at various distances from the axis, are of the same order of magnitude as those obtained by the present calculation for the incipient turbulence.

Chapter II

ENERGY DISTRIBUTION

7. The kinetic energy of the disturbance motion.

Having established the amplitude distribution for the two neutral vibrations, the energy of the disturbance motion can be computed. The total kinetic energy of the disturbance motion in a layer of unit height, which, in x direction, extends over a wave length λ and in y direction from the wall to infinity is

$$E = \frac{\rho}{2}\int_{x=0}^{\lambda}\int_{y=0}^{\infty}\left(u'^2 + v'^2\right)dx\,dy$$

$$= \frac{\rho}{2}\,\frac{\lambda\delta}{2}K^2U_m^2\int_0^{\infty}\left[\varphi_r'^2 + \varphi_i'^2 + \alpha^2\left(\varphi_r^2 + \varphi_i^2\right)\right]d(y/\delta) \qquad (34)$$

The energy dE of the secondary motion in a strip of width dy and length λ is accordingly

$$\frac{dE}{dy} = \frac{\rho}{2}\,\frac{\lambda}{2}U_m^2K^2\left\{\varphi_r'^2 + \varphi_i'^2 + \alpha^2\left(\varphi_r^2 + \varphi_i^2\right)\right\} \qquad (35)$$

Besides,

$$0.533\,\frac{\delta}{E_o}\,\frac{dE}{dy} = \frac{\overline{u'^2} + \overline{v'^2}}{U_m^2} \qquad (36)$$

E_o is the basic-flow energy in a layer of unit height, length λ, and width δ (compare equation (32)). Figure 7 shows the dimensionless energy distribution by equations (35) and (36). The energy is strongly concentrated near the critical layer. To obtain the total energy E, the integral (34) must be evaluated. Dividing it in two parts with the limits $0 \leqq y/\delta \leqq 1.015$ and $1.015 \leqq y/\delta < \infty$, the first portion is obtained by graphical integration based on the computed amplitude distribution. The second portion is obtained analytically by (29), namely

$$\int_{y/\delta=1.015}^{\infty} \left[\varphi_r'^2 + \varphi_i'^2 + \alpha^2(\varphi_r^2 + \varphi_i^2)\right] d(y/\delta) = \alpha\delta C^{*2} e^{-2.03\alpha\delta}$$

The results of the evaluation are given in table 1.

Now the energy of the disturbance motion is compared with the basic-flow energy E_0 in the space of unit height and surface area $\lambda \times \delta$. It is, by equation (7a)

$$E_0 = \frac{\rho}{2} \int_{x=0}^{\lambda} \int_{y=0}^{\delta} U(y)\,dx\,dy = 0.533\, \frac{\rho}{2} U_m^2 \lambda\delta \tag{37}$$

Hence, for the ratio of energy of the secondary motion to the energy of the basic flow E/E_0 the values presented in table 1 are obtained.

8. The energy balance of the disturbance motion.

Consider the time variation of the secondary-motion energy of a particle that moves with the basic flow, hence

$$\frac{D}{Dt}\left\{\frac{\rho}{2}(u'^2 + v'^2)\right\} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left\{\frac{\rho}{2}(u'^2 + v'^2)\right\} \tag{38}$$

For stable disturbances, the total change of energy of the secondary-motion is $\frac{\rho}{2}\iiint \frac{D}{Dt}(u'^2 + v'^2)\,dv < 0$, for unstable disturbances > 0, and for neutral disturbances $= 0$, the integration extending over the entire range of the particular flow. Participating on the variation of the secondary-motion energy are: first, the transfer of kinetic energy from the primary to the secondary flow, or vice versa; second, the pressure variation; and, third, dissipation. For neutral vibrations, the total energy balance is not only equal to zero for the entire space in question, but for every point y of the cross section, the energy increase per vibration period $T = 2\pi/\beta_r$ is also equal to zero. This is easily confirmed in the following manner: It is

$$\frac{\rho}{2}\int_{t=0}^{T} \frac{D}{Dt}(u'^2 + v'^2)dt$$

$$= \frac{\rho}{2}\int_{t=0}^{T} \frac{\partial}{\partial t}(u'^2 + v'^2)dt + \frac{\rho}{2} U\int_{t=0}^{T} \frac{\partial}{\partial x}(u'^2 + v'^2)dt$$

$$= \frac{\rho}{2}\left[u'^2 + v'^2\right]_0^T + \rho U\int_0^T \left(u' \frac{\partial u'}{\partial x} + v' \frac{\partial v'}{\partial x}\right)dt = 0$$

The first term disappears by reason of the periodicity of u' and v'. The same holds true when the last term for u' and v' is entered according to equation (30). Thus, the energy increase per vibration period T is equal to zero at every point x, y for a neutral vibration.

It is interesting to see how the several factors enumerated above participate on the energy conversion in a specific case. For both specific cases of neutral disturbance the calculation of the energy is carried out for a plane basic flow and a plane disturbance motion according to Lorentz (reference 12)

$$\frac{D}{Dt}\left\{\frac{\rho}{2}(u'^2 + v'^2)\right\} = -\rho u'v' \frac{dU}{dy} - \left\{\frac{\partial(u'p')}{\partial x} + \frac{\partial(v'p')}{\partial y}\right\}$$

$$- \mu\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)^2 + \mu\left\{\frac{\partial}{\partial x}(v'\zeta') - \frac{\partial}{\partial y}(u'\zeta')\right\}$$

where

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}, \ \mu = \text{coefficient of viscosity}$$

The first term gives the transfer of energy from the primary to the secondary flow, the second gives the contribution resulting from the pressure variations, and the third and fourth terms, the loss of energy by dissipation. After integration of this term with respect to y over the total width of the laminar flow from $y = 0$ to $y = \infty$ and with respect to x over a wave length λ of the disturbance, the second and fourth terms disappear, since u' and v' disappear for $y = 0$ and $y = \infty$ and with respect to x have the period λ. Thus, the growth of the energy per unit time in a layer of unit height and base area $0 < y < \infty$, $0 < x < \lambda$ is:

$$\frac{DE}{Dt} = -\rho \int_{x=0}^{\lambda} \int_{y=0}^{\infty} u'v' \frac{dU}{dy}\,dxdy - \mu \int_{y=0}^{\lambda} \int_{y=0}^{\infty} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)^2 dxdy \tag{39}$$

The first integral gives the total energy passing from the primary to the secondary motion; the second, the total dissipation. The portion of the energy due to pressure variation is removed by the integration. The two energy portions for the two neutral disturbances are evaluated. Through substitution of (30), followed by integration with respect to x, we find

$$\frac{2}{\lambda}\frac{DE}{Dt} = -\rho\alpha K^2 U_m^{\ 2} \int_{\eta=0}^{\infty} \left(\varphi_r'\varphi_i - \varphi_r\varphi_i'\right) \frac{dU}{dy}\,dy - \mu U_m^{\ 2} \int_{\eta=0}^{\infty} \left\{\left(\varphi_r'' - \alpha^2\varphi_r\right)^2 + \left(\varphi_i'' - \alpha^2\varphi_i\right)^2\right\} dy$$

or

$$\frac{\lambda}{U_m}\frac{DE}{Dt} = 2\pi K^2(e_1 + e_2)\,\lambda\delta\,\frac{\rho}{2}\,U_m^{\ 2} \tag{40}$$

e_1 and e_2 denoting the dimensionless energy integrals

$$e_1 = -\int_0^{\infty} \left(\varphi_r'\varphi_i - \varphi_r\varphi_i'\right) \frac{d(U/U_m)}{d(y/\delta)}\,d(y/\delta) = \int_0^{\infty} e_1'\,d(y/\delta) \tag{41a}$$

$$e_2 = -\frac{1}{\alpha R}\int_0^\infty \left\{(\varphi_r'' - \alpha^2\varphi_r)^2 + (\varphi_i'' - \alpha^2\varphi_i)^2\right\} d(y/\delta) = \int_0^\infty e_2' d(y/\delta) \quad (41b)$$

To find the energy change of the disturbance motion in the vibration period $T(T = \lambda/c_r,\ c_r$ = phase velocity), this energy change is referred to the total kinetic energy E of the secondary motion which is given by equation (34). From (40) follows then the specific energy change of the disturbance motion as

$$\frac{T}{E_0}\frac{DE}{Dt} = \frac{U_m}{c_r}\frac{2\pi}{0.533Z}\int_0^\infty (e_1' + e_2')d(y/\delta) \quad (42)$$

where $Z = 0.432$ and $Z = 0.810$ for the first and second neutral vibration, respectively, while $U_m/c_r = 2.86$ for both neutral vibrations.

The local energy transfer from primary to secondary motion (1) and the local dissipation (2) for the two neutral vibrations is then

$$\frac{\delta}{E}\frac{d}{dy}(\Delta E)_{1,2} = 78.0e_{1,2}' \quad \text{or} \quad = 41.6e_{1,2}' \quad (43a, b)$$

when

$$T\frac{DE}{Dt} = \Delta E$$

The values of $e_{1,2}'$ (equations 41a, b) can be obtained (table 8) on the basis of the computed amplitude distribution for both neutral vibrations. Figure 9 represents the local energy conversion. The dissipation in wall proximity is seen to be extremely great, while the critical layer is of no particular importance for the dissipation. But the energy transfer from the primary to the secondary motion is greatest in the neighborhood of the critical layer, while at the wall and farther outside it is very small. The curve is similar to that of the correlation (fig. 8), as anticipated.

The graphical integration of e_1' and e_2' gives the values indicated in table 1. The energy balance $\frac{DE}{Dt} = 0$, or $e_1 + e_2 = 0$

for the neutral vibrations is therefore fulfilled with satisfactory approximation[5], and constitutes a very welcome check on the rather complicated solution of the characteristic value problem.

The total energy transferred in vibration period T from the primary to the secondary motion is

$$(\Delta E)_1/E = 78.0e_1 \quad \text{or} \quad 41.6e_1$$

and the total energy dissipated

$$(\Delta E)_2/E = 78.0e_2 \quad \text{or} \quad 41.6e_2$$

These figures are also shown in table 1. Thus, at the first neutral vibration, about half of the secondary-motion energy is destroyed by dissipation during one vibration; at the second neutral vibration, the energy conversion is only about half as great.

At the second neutral vibration, the vibration period is a little greater than at the first, that is, as is readily obtainable from the data of table 1, is

$$T_1 = \left(\frac{2\pi}{\beta_r}\right)_1 = 10.1 \times 10^4 \frac{\nu}{U_m^2}; \quad T_2 = \left(\frac{2\pi}{\beta_r}\right)_2 = 14.8 \times 10^4 \frac{\nu}{U_m^2}$$

To illustrate; For a plate flow in water with

$$\nu = 0.01 \text{ cm}^2\text{sec}^{-1}; \quad U_m = 20 \text{ cm sec}^{-1}$$

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[5]According to the present calculation, the dissipation for both vibrations is somewhat greater than the transfer of energy from the primary to the secondary flow. This is due to the fact that in the stability calculation only the dissipation of the friction solution φ_3 was taken into account, while the dissipation of the frictionless vibration (φ_1, φ_2) was ignored. But, in the energy equation, the dissipation of frictionless and frictional vibration was computed and is therefore a little greater. Thus, the "neutral vibrations" have, exactly computed, still a little damping, and the indifference curve (fig. 2) is, as a result, shifted a little toward the inside.

the periods of vibration are

$$T_1 = 2.50 \text{ sec}; \; T_2 = 3.70 \text{ sec}$$

Thus, vibrations of comparatively great periods are involved.

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TABLE 1

THE PARAMETERS OF THE TWO NEUTRAL VIBRATIONS OF THE PLATE FLOW

	First neutral vibration	Second neutral vibration
$\alpha\delta$	0.466	0.737
$(U_m\delta/\nu)$	2.62×10^3	6.08×10^3
$\beta_r\delta/U_m)$	.163	.258
c_r/U_m	.350	.350
y_k	.209	.209
U_k'	1.625	1.625
U_k''/U_k'	−.494	−.494
η_0	−2.63	−4.05
$1/\epsilon$	12.6	19.4
β_1	.0695 +.102i	−.0470 +.0276i
β_2	−.1526 −.0736i	.0368 −.0650i
D	1.374 +.200i	.395 +1.24i
φ_{1a}	.416	.435
φ_{1a}'	.040	.097
φ_{2a}	.211	.306
φ_{2a}'	−2.425	−2.240
Φ_{1a}	.234	.417
Φ_{2a}	−2.327	−2.014
C_2	.101	.207
φ_{10}'	1.005	1.011
φ_{20}'	−.046 +1.563i	−.114 +1.563i
C_3'	−1.00 −.157i	−.988 −.325i
C*	.706	1.075
K	.1454	.1166
$\int_0^{1.015} \left\{\varphi_r'^2 + \varphi_i'^2 + \alpha^2(\varphi_r^2 + \varphi_i^2)\right\} dy/\delta$	.371	.681
$\int_{1.015}^{\infty} \left\{\right\} dy/\delta$	.090	.183

TABLE 1

THE PARAMETERS OF THE TWO NEUTRAL VIBRATIONS OF THE PLATE FLOW – Concluded

	First neutral vibration	Second neutral vibration
$\frac{E}{E_0}$	$0.432K^2$	$0.810K^2$
$\frac{E}{E_0}$	0.00913	0.0110
$e_1 \times 10^3$	5.75	6.39
$e_2 \times 10^3$	–6.16	–7.10
$\frac{(\Delta E)_1}{E}$	0.447	0.265
$\frac{(\Delta E)_2}{E}$	–0.479	–0.294

TABLE 2

NEWLY CALCULATED VALUES OF β_1 AND D

$-\eta_o$	β_1	D
0	0.387 + 0.672i	0.672 − 0.387i
0.5	0.341 + 0.366i	0.770 − 0.380i
1.0	0.262 + 0.213i	0.892 − 0.350i
1.5	0.192 + 0.142i	1.023 − 0.261i
2.0	0.132 + 0.113i	1.202 − 0.135i
2.5	0.0822 + 0.1031i	1.358 + 0.124i
3.0	0.0332 + 0.0972i	1.397 + 0.519i
3.5	−0.0165 + 0.0782i	1.139 + 1.023i
4.0	−0.0465 + 0.0323i	0.493 + 1.214i

TABLE 3

THE PARTICULAR SOLUTIONS OF φ_3 AS FUNCTION OF η

[Compare equation (15)]

η	F_{1r}	F_{1i}	F_{2r}	F_{2i}	F_{1r}'	F_{1i}'	F_{2r}'	F_{2i}'
0	0	0	0	0	0	0	0	0
0.5	0.021	0.000	0.125	0.000	0.1250	0.0005	0.5000	0.0026
1.0	0.167	0.003	0.500	0.008	0.4998	0.0167	0.9992	0.0417
1.5	0.563	0.032	1.122	0.063	1.1186	0.1264	1.4865	0.2065
2.0	1.318	0.177	1.974	0.266	1.9368	0.5292	1.8988	0.6588
2.5	2.59	0.678	2.97	0.797	2.7503	1.5799	2.0223	1.5544
2.63	2.87	0.901	3.24	1.022	2.90	2.02	1.95	1.88
3.0	3.97	1.94	3.86	1.91	2.9208	3.6995	1.3354	2.9269
3.5	5.07	4.55	4.02	3.72	0.9512	6.8796	−1.0850	4.2113
4.0	4.20	8.70	2.35	5.77	−5.6379	9.2445	−6.1155	3.3059
4.05	3.84	9.13	2.00	5.89	−6.62	9.20	−6.76	2.90
	antisy.	sy.	sy.	antisy.	sy.	antisy.	antisy.	sy.

η	F_{1r}''	F_{1i}''	F_{2r}''	F_{2i}''	F_{1r}'''	F_{1i}'''	F_{2r}'''	F_{2i}'''
0	0	0	1	0	1	0	0	0
0.5	0.5000	0.0052	0.9999	0.0208	0.9998	0.0417	−0.0010	0.1250
1.0	0.9980	0.0833	0.9944	0.1666	0.9861	0.3331	−0.0333	0.4999
1.5	1.4661	0.4206	0.9368	0.5595	0.8442	1.1165	−0.2525	1.1072
2.0	1.7472	1.3108	0.6469	1.2939	0.1187	2.5541	−1.0533	1.8229
2.5	1.3100	3.0463	−0.3215	2.3124	−2.2818	4.3761	−3.0889	2.0809
2.63	0.941	3.64	−0.776	2.58				
3.0	−1.1161	5.4735	−2.7429	3.0209	−8.1612	4.7949	−6.8785	0.1417
3.5	−7.6314	6.7189	−7.2934	1.4618	−18.5245	−1.7400	−11.0222	−7.8189
4.0	−19.4902	0.6805	−12.5621	−6.6661	−27.2703	−26.6992	−7.4580	−26.7961
4.05	−20.94	−0.806	−12.92	−8.10				
	antisy.	sy.	sy.	antisy.	sy.	antisy.	antisy.	sy.

TABLE 4

THE FRICTION SOLUTION φ_3 AS FUNCTION

OF y/δ AND η

First Neutral Vibration

η	y/δ	$\frac{\varphi_{3r}''}{\varphi_{30}'}$	$\frac{\varphi_{3i}''}{\varphi_{30}'}$	$\frac{\varphi_{3r}'}{\varphi_{30}'}$	$\frac{\varphi_{3i}'}{\varphi_{30}'}$	$\frac{\varphi_{3r}}{\varphi_{30}'}$	$\frac{\varphi_{3i}}{\varphi_{30}'}$
-2.63	0	-6.40	7.65	1	0	-0.111	-0.0159
-2	.050	-5.67	.794	.665	.188	-.069	-.0099
-1	.130	-3.04	-1.801	.329	.106	-.029	.0029
0	.209	-1.92	-.932	.136	-.012	-.009	.0061
1	.288	-.995	.113	.019	-.040	-.001	.0035
2	.368	-.202	.290	-.025	-.018	0	.0010
3	.447	.063	.076	-.027	-.003	0	.0005
4	.577	-.076	-.012	-.020	.012	0	.0006

Second Neutral Vibration

η	y/δ	$\frac{\varphi_{3r}''}{\varphi_{30}'}$	$\frac{\varphi_{3i}''}{\varphi_{30}'}$	$\frac{\varphi_{3r}'}{\varphi_{30}'}$	$\frac{\varphi_{3i}'}{\varphi_{30}'}$	$\frac{\varphi_{3r}}{\varphi_{30}'}$	$\frac{\varphi_{3i}}{\varphi_{30}'}$
-4.05	0	-17.7	34	1	0	-0.0204	-0.0639
-3.5	.029	-17.6	8.22	.460	.519	.0001	-.0554
-3	.054	-9.71	-3.08	.106	.534	.0070	-.0415
-2.63	.074	-4.91	-4.67	-.030	.455	.0076	-.0322
-2	.105	-.272	-3.88	-.103	.309	.0051	-.0202
-1	.157	1.378	-2	-.058	.166	.0005	-.0091
0	.209	.717	-1.260	0	.084	-.0008	-.0035
1	.261	-.019	-.679	.016	.035	-.0002	-.0012
2	.313	-.194	-.155	.007	.013	0	0
3	.361	-.058	.019	0	.012	0	0
4	.415	.039	0	-.005	.014	0	0

TABLE 5

THE FRICTION CORRECTION φ_3 IN THE

INTERMEDIATE LAYER AS FUNCTION OF η

η	$G''(\eta)$	$H''(\eta)$	$G'(\eta)$	$H'(\eta)$
0	0	1.285	−0.774	−1.570
.5	.443	1.165	−.659	−.953
1	.746	.857	−.354	−.448
1.5	.839	.483	.051	−.118
2	.767	.160	.458	.035
2.5	.589	−.018	.798	.073
3	.438	−.072	1.056	.048
3.5	.325	−.055	1.243	.013
4	.250	0	1.386	0

$$G''(-\eta) = -G''(\eta);\ H''(-\eta) = H''(\eta)$$
$$G'(\eta) = G'(\eta);\ H'(-\eta) = -\pi - H'(\eta)$$

TABLE 6

DISTRIBUTION OF AMPLITUDES φ_r, φ_i, φ_r', φ_i', φ_r'', φ_i''

AS FUNCTION OF y/δ

First Neutral Vibration

y/δ	φ_r	φ_i	φ_r'	φ_i'	φ_r''	φ_i''
0	0	0	0	0	7.885	−6.615
.050	.011	−.0040	.411	−.135	6.235	−.003
.090	.032	−.0080	.620	−.115	4.580	.770
.130	.059	−.0112	.782	−.025	3.200	1.717
.170	.090	−.0111	.896	.042	2.355	1.240
.209	.127	−.0088	.976	.069	1.778	.434
.250	.166	−.0063	.986	.070	−.112	−.220
.290	.203	−.0041	.975	.058	−.699	−.576
.370	.276	−.0010	.831	.018	−1.543	−.347
.451	.335	0	.670	.005	−1.778	−.015
.531	.380	0	.510	−.003	−1.641	−.016
.612	.414	0	.361	0	−1.606	0
.693	.438	0	.237	0	−1.514	0
.774	.453	0	.118	0	−1.431	0
.854	.458	0	.007	0	−1.364	0
.935	.455	0	−.101	0	−1.303	0
1.015	.445	0	−.205	0	−1.252	0

TABLE 6

DISTRIBUTION OF AMPLITUDES φ_r, φ_i, φ_r', φ_i', φ_r'', φ_i''

AS FUNCTION OF y/δ – Concluded

Second Neutral Vibration

y/δ	φ_r	φ_i	φ_r'	φ_i'	φ_r''	φ_i''
0	0	0	0	0	28.54	–27.89
.029	.010	–.0038	.720	–.337	20.27	–2.30
.054	.033	–.0114	1.104	–.237	9.16	6.35
.074	.057	–.0140	1.245	–.113	4.25	6.28
.105	.096	–.0148	1.333	.052	.478	3.60
.157	.168	–.0086	1.350	.129	.558	–.27
.209	.236	–.0041	1.306	.076	–1.218	–1.54
.250	.287	.0005	1.203	.014	–2.967	–1.31
.290	.333	.0002	1.091	–.016	–2.961	–.334
.370	.409	0	.827	–.016	–2.86	.144
.451	.469	0	.607	0	–2.58	0
.531	.507	0	.409	0	–2.21	0
.612	.532	0	.247	0	–1.93	0
.693	.547	0	.105	0	–1.72	0
.774	.550	0	–.026	0	–1.56	0
.854	.544	0	–.148	0	–1.44	0
.935	.527	0	–.260	0	–1.32	0
1.015	.508	0	–.367	0	–1.23	0

TABLE 7

THE MEAN FLUCTUATION VELOCITIES $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, THE KINETIC ENERGY OF THE DISTURBANCE MOTION $\overline{u'^2} + \overline{v'^2}$, AND THE CORRELATION COEFFICIENT k AS FUNCTION OF y/δ. [EQUATIONS (31), (32), (33)]

First Neutral Vibration

y/δ	$10 \frac{\sqrt{\overline{u'^2}}}{U_m}$	$10^2 \frac{\sqrt{\overline{v'^2}}}{U_m}$	$10^2 \frac{\overline{u'^2} + \overline{v'^2}}{U_m^2}$	$-k$
0	0	0	0	0
.050	.445	.0546	.198	.032
.090	.647	.158	.420	.061
.130	.804	.287	.648	.155
.170	.922	.434	.854	.169
.209	1.005	.608	1.019	.140
.250	1.016	.795	1.040	.109
.290	1.003	.973	1.019	.064
.370	.854	1.322	.748	.022
.451	.689	1.605	.501	.0075
.531	.524	1.820	.309	0
.612	.371	1.984	.177	0
.693	.244	2.098	.104	0
.774	.121	2.170	.062	0
.854	.007	2.193	.048	0
.935	.103	2.180	.059	0
1.015	.211	2.108	.090	0
1.1	.203	2.027	.082	0
1.2	.193	1.935	.075	0
1.3	.185	1.849	.068	0
1.4	.176	1.763	.062	0
1.5	.168	1.681	.057	0

TABLE 7

THE MEAN FLUCTUATING VELOCITIES $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$,

THE KINETIC ENERGY OF THE DISTURBANCE MOTION $\overline{u'^2} + \overline{v'^2}$,

AND THE CORRELATION COEFFICIENT k AS FUNCTION OF y/δ.

EQUATIONS (31), (32), (33) — Concluded

Second Neutral Vibration

y/δ	$10\,\dfrac{\sqrt{\overline{u'^2}}}{U_m}$	$10^2\,\dfrac{\sqrt{\overline{v'^2}}}{U_m}$	$10^2\,\dfrac{\overline{u'^2} + \overline{v'^2}}{U_m^2}$	$-k$
0	0	0	0	0
.029	.661	.0655	.433	0
.054	.937	.214	.869	.121
.074	1.038	.359	1.068	.170
.105	1.108	.595	1.218	.191
.157	1.127	1.028	1.262	.146
.209	1.086	1.445	1.190	.076
.250	.998	1.755	1.019	.010
.290	.906	2.04	.854	−.015
.370	.686	2.50	.528	−.019
.451	.504	2.87	.332	0
.531	.340	3.10	.208	0
.612	.205	3.26	.146	0
.693	.087	3.35	.118	0
.774	.022	3.37	.117	0
.854	.123	3.33	.124	0
.935	.216	3.22	.149	0
1.015	.305	3.11	.187	0
1.1	.292	2.925	.171	0
1.2	.272	2.72	.146	0
1.3	.252	2.53	.126	0
1.4	.234	2.35	.109	0
1.5	.218	2.18	.093	0

TABLE 8

THE LOCAL ENERGY CONVERSION 1) = TRANSFER FROM PRIMARY TO SECONDARY MOTION, 2) = DISSIPATION EQUATIONS (41) AND (43)

First Neutral Vibration

y/δ	$e_1' \times 10^3$	$e_2' \times 10^3$	$\frac{\delta}{E}\frac{d}{dy}(\Delta E)_1$	$\frac{\delta}{E}\frac{d}{dy}(\Delta E)_2$
0	0	86.9	0	-6.78
.050	.268	31.8	.021	-2.48
.090	2.150	17.6	.168	-1.37
.130	12.25	10.8	.955	-.84
.170	23.25	5.73	1.814	-.45
.209	28	2.64	2.18	-.21
.250	27.30	.06	2.13	0
.290	22.85	.72	1.78	-.05
.370	7.49	2.20	.584	-.17
.451	1.885	2.80	.147	-.22
.531	-1.102	2.43	-.086	-.19
.612	0	2.36	0	-.18
.693	0	2.12	0	-.17
.774	0	1.92	0	-.15
.854	0	1.75	0	-.14
.935	0	1.61	0	-.13
1.015	0	1.50	0	-.12

TABLE 8

THE LOCAL ENERGY CONVERSION 1) = TRANSFER FROM PRIMARY TO SECONDARY MOTION, 2) = DISSIPATION [EQUATIONS (41) AND (43)] – Concluded

Second Neutral Vibration

y/δ	$e_1' \times 10^3$	$e_2' \times 10^3$	$\frac{\delta}{E}\frac{d}{dy}(\Delta E)_1$	$\frac{\delta}{E}\frac{d}{dy}(\Delta E)_2$
0	0	320	0	–13.31
.029	1	83.6	–.044	–3.48
.054	7.84	24.3	.326	–1.01
.074	18.45	11.15	.768	–.464
.105	41.5	2.61	1.726	–.109
.157	55.6	.04	2.310	–.002
.209	37.5	.840	1.560	–.035
.250	5.22	2.31	.217	–.096
.290	7.97	2.01	–.331	–.084
.370	–8.44	1.91	–.351	–.079
.451	0	1.80	0	–.075
.531	0	1.38	0	–.057
.612	0	1.01	0	–.042
.693	0	.819	0	–.034
.774	0	.695	0	–.029
.854	0	.608	0	–.025
.935	0	.520	0	–.022
1.015	0	.458	0	–.019

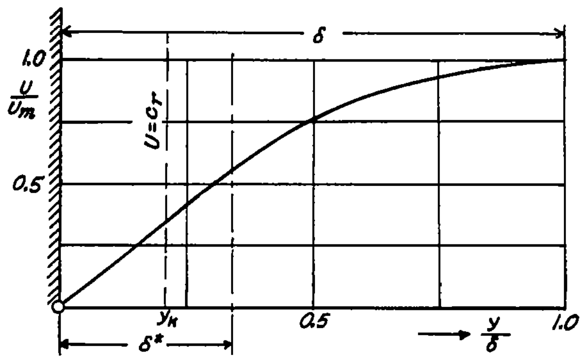

Figure 1.- Laminar flow past the plate.

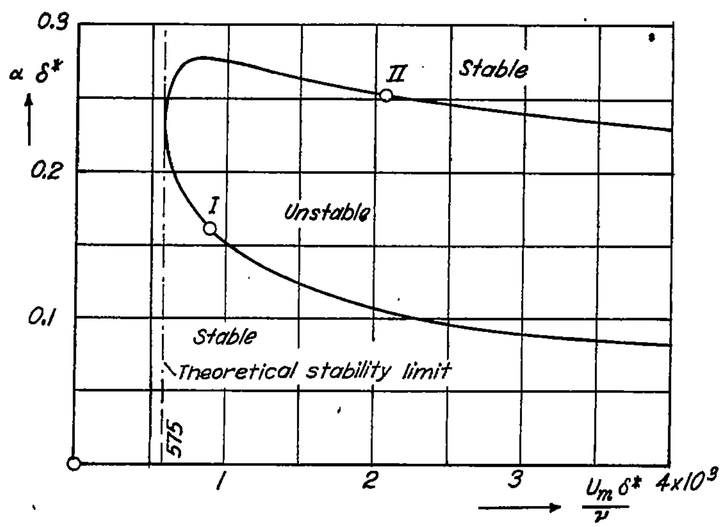

Figure 2.- The zone of the stable and unstable disturbances of plate flow. I = first neutral vibration. II = second neutral vibration.

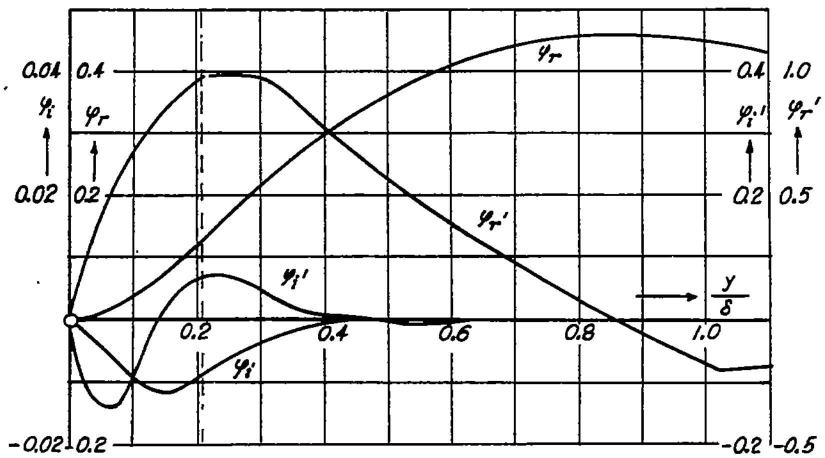

Figure 3.- Real and imaginary part $\varphi_r, \varphi_i, \varphi_r', \varphi_i'$ of the amplitude of disturbance motion plotted against wall distance for the first neutral vibration.

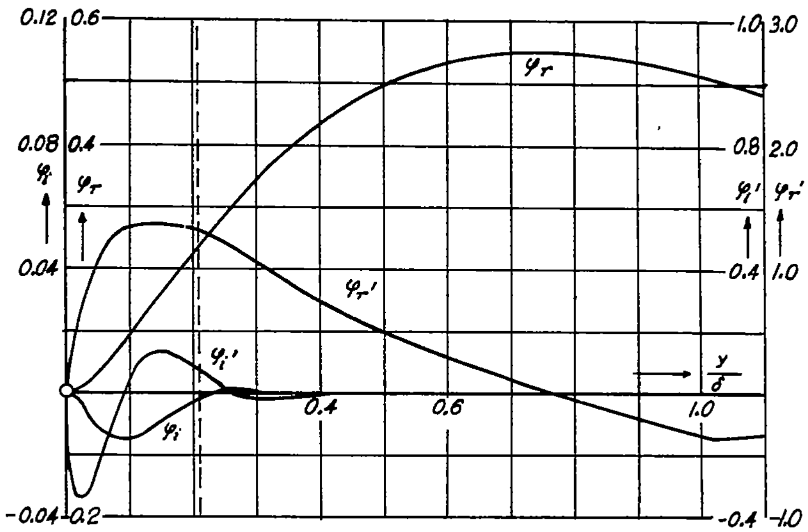

Figure 4.- Real and imaginary part $\varphi_r, \varphi_i, \varphi_r', \varphi_i'$ of the amplitude of disturbance motion plotted against wall distance for the second neutral vibration.

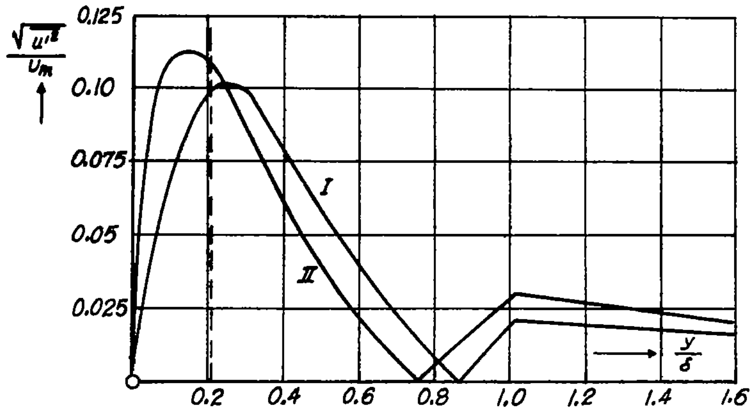

Figure 5.- The mean fluctuating velocity in the x direction $\sqrt{\overline{u'^2}}/U_m$ plotted against the wall distance for both neutral vibrations.

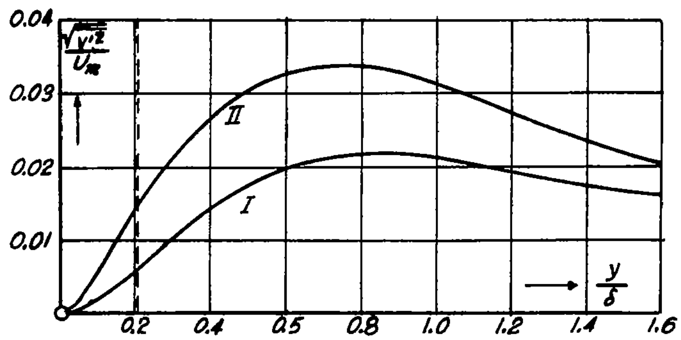

Figure 6.- The mean fluctuating velocity in the y direction $\sqrt{\overline{v'^2}}/U_m$ plotted against the wall distance for both neutral vibrations.

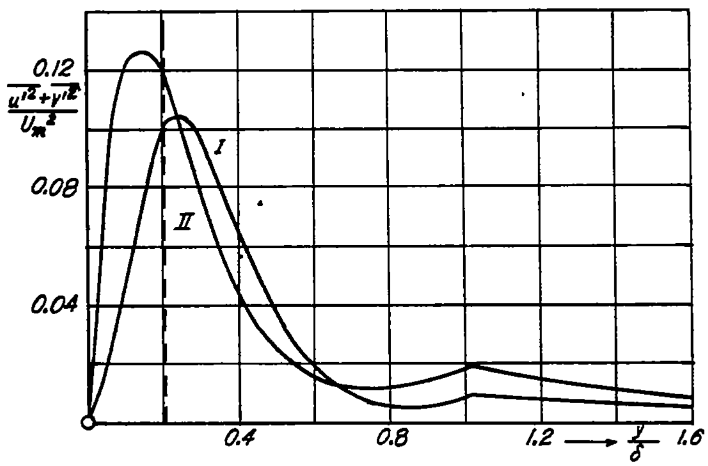

Figure 7.- The mean kinetic energy of the disturbance motion $\overline{u'^2} + \overline{v'^2} / U_m^2$ plotted against the wall distance for both neutral vibrations.

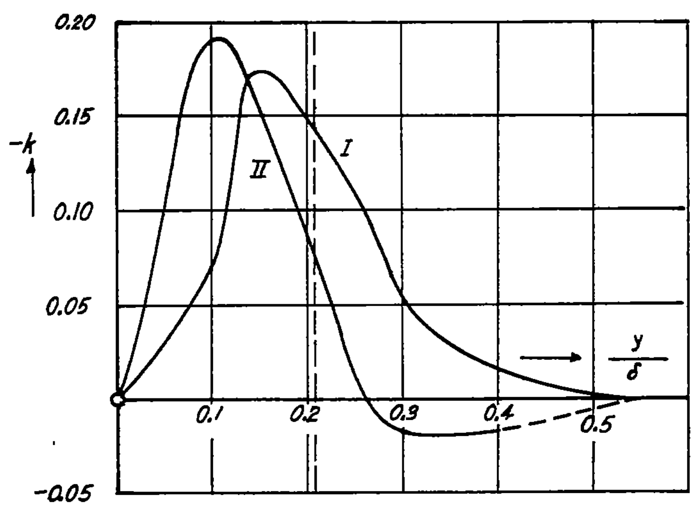

Figure 8.- The correlation coefficient $k = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2} \cdot \overline{v'^2}}}$ plotted against y/δ for both neutral vibrations.

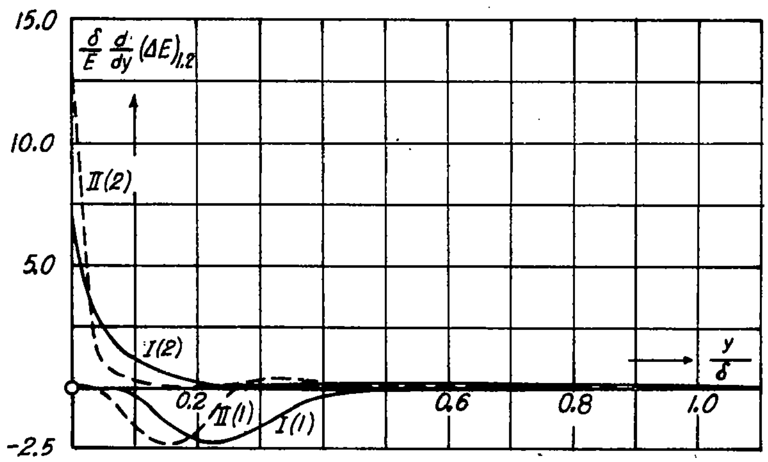

Figure 9.- The local energy conversion of the secondary motion for the first and second neutral vibration. I (1), II (1) = energy transfer from primary to secondary flow; I (2), II (2) = dissipation.

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